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Phase dynamics in the complex Ginzburg–Landau equation

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Abstract

For $\alpha\beta > -1$, stable time periodic solutions $A(X,T) = A_q e^{iqX + i\omega_q T}$ are the locally preferred planform for the complex Ginzburg–Landau equation

$$\partial_T A = A + (1 + i\alpha)\partial_X^2 A - (1 + i\beta)A|A|^2.$$

In order to describe the spatial global behavior, an evolution equation for the local wave number q can be derived formally. The local wave number q satisfies approximately a conservation law $\partial_\tau q = \partial_\xi h(q)$. It is the purpose of this paper to explain the extent to which the conservation law is valid by proving estimates for this formal approximation.

Keywords: Approximation; complex Ginzburg-Landau; conservation law; pattern formation

1. Introduction

The (normalized) complex Ginzburg–Landau equation

$$\partial_T A = A + (1 + i\alpha)\partial_X^2 A - (1 + i\beta)A|A|^2$$

with $X \in \mathbb{R}$, $T \geqslant 0$, $A(X,T) \in \mathbb{C}$, and coefficients $\alpha, \beta \in \mathbb{R}$, is an universal amplitude equation which is derived by multiple scaling analysis in order to describe bifurcating solutions in pattern forming systems close to the threshold of the first instability. The

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amplitude A describes slow modulations in space and time of the underlying bifurcating spatially periodic pattern. Examples of such pattern forming systems are reaction–diffusion systems, systems in nonlinear optics, or hydrodynamical stability problems, for example Rayleigh–Bénard convection or the Taylor–Couette problem. A mathematical theory of the reduction to the Ginzburg–Landau equation has been developed by several authors (cf. [CE90,Me98,Me99,Me00,Schn94,vH91]). It is nowadays a well-established mathematical tool which can be used to obtain new mathematical results (cf. [Schn99]), including global existence results and upper-semicontinuity of attractors.

The complex Ginzburg–Landau equation possesses a family of time-periodic solutions

$$A(X,T) = r_q e^{iqX+i\omega_q T+i\phi_0} = A_{\mathrm{per}}[q,\phi_0](X,T)$$

with $q, \phi_0, r_q, \omega_q \in \mathbb{R}$. For $\alpha\beta \geqslant -1$, these solutions are spectrally stable and hence are the preferred planform locally in space. In order to describe the global behavior in space, an evolution equation for the local wave number q can be derived. Allowing q to vary slowly in time and space, we define

$$A_{\mathrm{per}}[\psi(\delta X,\delta T),\phi_0](X,T) = r_{\psi(\delta X,\delta T)} \exp\left(i\int_0^X \psi(\delta X',\delta T))\, dX' + i\omega_0 T + i\phi_0 \right)$$

with $0<\delta\ll 1$ a small perturbation parameter, where ψ satisfies the conservation law

$$\partial_\tau \psi = \partial_\xi h(\psi) \tag{1}$$

with $\tau = \delta T$, $\xi = \delta X$, and $h:\mathbb{R}\to\mathbb{R}$ a smooth function. Note that ω_q is evaluated at $q=0$, in contrast to r_q which is evaluated at $q=\psi$. It is the purpose of this paper to explain to which extent this formal approximation is valid by proving estimates between the formal approximation $A_{\mathrm{per}}[\psi(\delta X,\delta T),\phi_0](X,T)$ and exact solutions $A = A(X,T)$ of the complex Ginzburg-Landau equation.

Although the spatially periodic pattern are only spectrally stable [Eck65] for $\alpha\beta \geqslant -1$ the approximation property also holds in the unstable case, i.e. also for $\alpha\beta < -1$. However, the approximation property becomes worse for $\alpha\beta \to -\infty$.

It turns out that we cannot expect validity uniformly for all $X\in\mathbb{R}$. Instead, we show that the conservation law approximation is uniformly valid for all $X\in I_\delta$ where I_δ is an interval of length $\mathcal{O}(\delta^{-r})$. Here, $r>0$ is arbitrary but fixed depending on the chosen rate of approximation. Moreover, we have to allow for a global phase $\exp(i\phi(0,T))$.

It is not obvious a priori that an approximation result for the conservation law (1) holds. There are a number of counterexamples of amplitude equations which are derived formally in a correct way, but do not describe the dynamics in the original system in a correct way [Schn95].

The difficulty in justifying the conservation law for the Ginzburg–Landau equation is the time scale $\mathcal{O}(1/\delta)$. Since the solutions in consideration are of order $\mathcal{O}(1)$, a simple application of Gronwall's inequality would only give a time scale

$\mathcal{O}(1)$. Moreover, the Ginzburg–Landau equation in polar coordinates is quasi-linear, the lowest order linear terms do not possess any smoothing properties, and the smallness of the lowest order nonlinear terms is due to derivatives.

Hence, the approximation property is proved in a scale of Banach spaces consisting of functions analytic in a strip of the complex plane.

Our approximation result allows us to find the dynamics of the conservation law in the complex Ginzburg–Landau equation. Moreover, the Ginzburg–Landau equation approximates more complicated pattern forming systems like the Taylor–Couette problem, close to the first instability, and so we can find the dynamics of the conservation law in these more complicated systems, too. The dynamics of scalar conservation laws can be computed explicitly with the help of the method of characteristics.

Away from the threshold of the first instability, conservation laws for the evolution of the local wave number can be derived in order to describe spatial and temporal modulations of the fully developed spatially periodic pattern (cf. [HK77]). It is the purpose of further research to justify the conservation laws also away from the threshold of the first instability.

Other amplitude equations for the evolution of the local wave number of stable and unstable planforms in the Ginzburg–Landau equation have been considered in [Ber88,vH95]. For instance, by a different scaling Burgers equation

$$\partial_\tau \psi = (\alpha\beta + 1)\partial_\xi^2 \psi + (\beta - \alpha)\partial_\xi \psi^2 \tag{2}$$

can be derived. For some details see Remark 3.15. See [DSSS03] for an approximation result.

The plan of the paper is as follows. In Section 2 we derive the conservation law by introducing polar coordinates $A = re^{i\phi}$ and writing $\psi = \partial_X \phi$. In Section 3 we prove estimates which hold uniformly in space for the variables (r, ψ). In Section 4 we go back to the original A-variable which leads to the result that estimates which hold uniformly in space cannot be expected for the approximation of A. In Section 5 we explain the consequences of our result for the Taylor–Couette problem.

We note that the alternative approach of [Me98,Me99], discussed briefly in Remark 3.15, shows that the derivation of the conservation law (1) and simultaneously the Burgers equation (2) can be made exact for a certain class of solutions if derivative terms of all orders are included (so that Eqs. (1) and (2) are combined into a pseudo-differential equation).

Though the situation is formally very similar to that in [MS03], where for $\alpha = \beta = 0$ the associated phase diffusion equation has been justified, the rigorous arguments, especially in Section 3, are quite different. In [MS03] an optimal regularity argument has been applied. In the present paper the smoothing properties of the linear operator cannot be used and so a Cauchy–Kowalevskaya argument has to be used.

Notation: Throughout this paper we assume $0 < \delta \ll 1$. Many different constants are denoted with the same symbol C.

2. Derivation of the conservation law for the complex Ginzburg–Landau equation

As already said, the purpose of this paper is to justify the conservation law describing the evolution of the wave number q of the spatially locally preferred planform for the complex Ginzburg–Landau equation

$$\partial_T A = A + (1 + i\alpha)\partial_X^2 A - (1 + i\beta)A|A|^2 \tag{3}$$

with $\alpha, \beta \in \mathbb{R}$, $X \in \mathbb{R}$, $T \geqslant 0$, and $A(X, T) \in \mathbb{C}$. This equation possesses a family of time-periodic solutions

$$A(X, T) = re^{iqX + i\omega T + i\phi_0} \tag{4}$$

with $r = r_q > 0$, $q, \phi_0 \in \mathbb{R}$, and $\omega = \omega_q \in \mathbb{R}$, as spatially locally preferred planform. Inserting (4) into (3) gives

$$i\omega r = r - (1 + i\alpha)q^2 r - (1 + i\beta)r^3$$

and so dividing by r and equating real and imaginary parts, we obtain

$$r = \sqrt{1 - q^2}, \quad \omega = -\beta + (\beta - \alpha)q^2.$$

In order to derive the conservation law for the evolution of the local wave number q we again introduce polar coordinates

$$A(X, T) = r(X, T)e^{i\phi(X,T)}$$

and obtain

$$\begin{aligned}
\partial_T r &= \partial_X^2 r + r - (\partial_X \phi)^2 r - 2\alpha(\partial_X r)(\partial_X \phi) - \alpha(\partial_X^2 \phi)r - r^3, \\
\partial_T \phi &= \partial_X^2 \phi + \frac{\alpha \partial_X^2 r}{r} - \alpha(\partial_X \phi)^2 + \frac{2(\partial_X r)(\partial_X \phi)}{r} - \beta r^2.
\end{aligned} \tag{5}$$

We are interested in the dynamics close to the family of time-periodic solutions and so we introduce as new origin the time-periodic solution given in polar coordinates by $(r, \phi) = (1, -\beta T)$. We introduce the deviations $(s, \tilde{\phi})$ by setting $r = 1 + s$ and $\phi = -\beta T + \tilde{\phi}$. They satisfy

$$\begin{aligned}
\partial_T s &= \partial_X^2 s - 2s - (\partial_X \tilde{\phi})^2 - (\partial_X \tilde{\phi})^2 s - 2\alpha(\partial_X s)(\partial_X \tilde{\phi}) - \alpha\partial_X^2 \tilde{\phi} - \alpha(\partial_X^2 \tilde{\phi})s - 3s^2 - s^3, \\
\partial_T \tilde{\phi} &= \partial_X^2 \tilde{\phi} + \alpha \frac{\partial_X^2 s}{1 + s} - \alpha(\partial_X \tilde{\phi})^2 + \frac{2(\partial_X s)(\partial_X \tilde{\phi})}{1 + s} - 2\beta s - \beta s^2.
\end{aligned}$$

We can replace the equation for the phase $\tilde{\phi}$ by an equation for the local wave number $\psi = \partial_X \tilde{\phi}$ to obtain

$$\begin{aligned}
\partial_T s &= \partial_X^2 s - 2s - \psi^2 - \psi^2 s - 2\alpha(\partial_X s)\psi \\
&\quad - \alpha \partial_X \psi - \alpha(\partial_X \psi)s - 3s^2 - s^3, \\
\partial_T \psi &= \partial_X^2 \psi + \partial_X \left(\frac{\alpha \partial_X^2 s}{1+s} - \alpha\psi^2 + \frac{2(\partial_X s)\psi}{1+s} - 2\beta s - \beta s^2 \right).
\end{aligned} \tag{6}$$

The linearized system

$$\begin{aligned}
\partial_T s &= \partial_X^2 s - 2s - \alpha \partial_X \psi, \\
\partial_T \psi &= \partial_X^2 \psi + \alpha \partial_X^3 s - 2\beta \partial_X s
\end{aligned}$$

possesses solutions $(s, \psi) = (s_k, \psi_k) e^{ikx + \mu(k)t}$. For $k = 0$ we have $\mu_1(0) = 0$ and $\mu_2(0) = -2 < 0$. The negative eigenvalue $\mu_2(0) = -2$ corresponds to the s component and so we expect s to be slaved by ψ which will behave diffusively for $\alpha\beta > -1$ (cf. [vH95]). In order to derive the conservation law we make the long wave ansatz

$$\psi = \check{\psi}(\delta X, \delta T) \quad \text{and} \quad s = \check{s}(\delta X, \delta T)$$

and obtain

$$\begin{aligned}
\delta \partial_\tau \check{s} &= \delta^2 \partial_\xi^2 \check{s} - 2\check{s} - \check{\psi}^2 - \check{\psi}^2 \check{s} - 2\delta\alpha(\partial_\xi \check{s})\check{\psi} \\
&\quad - \delta\alpha \partial_\xi \check{\psi} - \delta\alpha(\partial_\xi \check{\psi})\check{s} - 3\check{s}^2 - \check{s}^3, \\
\partial_\tau \check{\psi} &= \delta \partial_\xi^2 \check{\psi} + \partial_\xi(-2\beta\check{s} - \alpha\check{\psi}^2 - \beta\check{s}^2) + \partial_\xi \left(\frac{\delta^2 \alpha \partial_\xi^2 \check{s}}{1+\check{s}} + \frac{2\delta(\partial_\xi \check{s})\check{\psi}}{1+\check{s}} \right),
\end{aligned} \tag{7}$$

where $\tau = \delta T$, $\xi = \delta X$. Neglecting terms of order $O(\delta)$ and higher gives

$$0 = -2\check{s} - \check{\psi}^2 - \check{\psi}^2 \check{s} - 3\check{s}^2 - \check{s}^3, \tag{8}$$

$$\partial_\tau \check{\psi} = \partial_\xi(-2\beta\check{s} - \alpha\check{\psi}^2 - \beta\check{s}^2). \tag{9}$$

For small $\check{\psi}$ the first equation (8) can be solved uniquely by the implicit function theorem, so there exists a smooth even function $s^* : \mathbb{R} \to \mathbb{R}$ such that $\check{s} = s^*(\check{\psi})$.

Inserting this into the second equation (9) gives the conservation law

$$\partial_\tau \check{\psi} = \partial_\xi(-2\beta s^*(\check{\psi}) - \alpha\check{\psi}^2 - \beta s^*(\check{\psi})^2) = \partial_\xi h(\check{\psi}), \tag{10}$$

where $h : \mathbb{R} \to \mathbb{R}$ is smooth.

Remark 2.1. The local existence and uniqueness of solutions of this scalar first-order equation is guaranteed by the method of characteristics or the Cauchy–Kowalevskaya theorem (cf. [Ov76]).

Remark 2.2. Suppose that instead we start with a general choice of basic periodic solution $r = r_q$, $\phi = qX + \omega_q t + \phi_0$, $q \in (-1, 1)$. Then the corresponding conservation law is given by

$$\partial_\tau \check{\psi} = \partial_\xi \tilde{h}(\check{\psi}, q), \tag{11}$$

where $\tilde{h}(\check{\psi}, q) = h(\check{\psi} + q)$.

To verify this, note that the deviations $\check{\psi}_q$ for (11) are related to the deviations $\check{\psi}_0$ for (10) by $\check{\psi}_q + q = \check{\psi}_0$. Hence $\partial_\tau \check{\psi}_q = \partial_\tau \check{\psi}_0 = \partial_\xi h(\check{\psi}_0) = \partial_\xi h(\check{\psi}_q + q) = \partial_\xi h(\check{\psi}_q, q)$ as required.

Remark 2.3. For each q, let $k \mapsto \mu_{1,2}(k, q)$ denote the smooth curves of eigenvalues corresponding to the Fourier wave numbers k for the linearization of (5) around the basic state $(r, \phi) = (\sqrt{1 - q^2}, qX + (-\beta + (\beta - \alpha)q^2)T)$. In particular, let $\mu_1(k, q)$ denote the critical curve for which $\mu_1(0, q) = 0$. Then we claim that the conservation law (10) must give at lowest order a linear conservation law $\partial_\tau \check{\psi} = h'(q)\partial_\xi \check{\psi}$ with drift coefficient

$$h'(q) = -i\partial_k \mu_1(0, q)$$

(which turns out by explicit calculation to be $h'(q) = 2(\beta - \alpha)q$).

First note that the linearization $L(q)$ of the right-hand side $\partial_X \tilde{h}(\psi, q)$ of (11) at $(0, q)$ must also have the eigenvalues $\mu_1(k, q)$—after taking into account the fact that higher order derivatives have been neglected in the derivation of (11). But

$$\partial_X \tilde{h}(\psi, q) = \partial_X h(q + \psi) = h'(q)\partial_X \psi + \mathcal{O}(|\psi|^2)$$

and so $L(q) = h'(q)\partial_X$. Equating $L(q)e^{ikX} = \mu_1(k, q)e^{ikx}$ modulo higher order derivatives yields $h'(q) = -i\partial_k \mu_1(0, q)$ as required.

Remark 2.4. It is common in the literature to consider generalized versions of the complex Ginzburg–Landau equation with more complicated nonlinearities. In general, terms of the form $A^{b_1}\bar{A}^{b_2}(\partial_X A)^{b_3}(\partial_X \bar{A})^{b_4}$ are permitted provided $b_1 - b_2 + b_3 - b_4 = 1$ and $b_3 + b_4$ is even. In this case, writing $A = re^{i\phi}$, $r = 1 + s$, $\psi = \partial_X \phi$, leads to a system of the following form in place of (6):

$$\begin{aligned}\partial_T s &= \partial_X^2 s - \alpha(\partial_X \psi)(1+s) + f(s, (\partial_X s)^2, \psi^2, (\partial_X s)\psi),\\ \partial_T \psi &= \partial_X^2 \psi + \alpha \partial_X(\partial_X^2 s/(1+s)) + \partial_X g(s, (\partial_X s)^2, \psi^2, (\partial_X s)\psi),\end{aligned} \qquad (12)$$

where f and g are smooth functions.

This structure can be verified as follows. The symmetries $X \mapsto -X$ and $A \mapsto e^{ic}A$ of the complex Ginzburg–Landau equation lead to the symmetries $X \mapsto -X$ and $\tilde{\phi} \mapsto \tilde{\phi} + c$ for the $(s, \tilde{\phi})$ equations. Hence, we obtain $\partial_T s = \partial_X^2 s - \alpha(\partial_X^2 \tilde{\phi})(1+s) + f(s, (\partial_X s)^2, (\partial_X \tilde{\phi})^2, (\partial_X s)(\partial_X \tilde{\phi}))$, $\partial_T \tilde{\phi} = \partial_X^2 \tilde{\phi} + \alpha(\partial_X^2 s)/(1+s) + g(s, (\partial_X s)^2, (\partial_X \tilde{\phi})^2, (\partial_X s)(\partial_X \tilde{\phi}))$. Writing $\psi = \partial_X \tilde{\phi}$ yields (12).

Now we can write $\tau = \delta T$ and $\xi = \delta X$ and neglecting small terms we reduce as before to the conservation law

$$\partial_\tau \check{\psi} = \partial_\xi g(s^*(\check{\psi}^2), 0, \check{\psi}^2, 0) = \partial_\xi h(\check{\psi}).$$

The estimates of this paper can be proved in a similar manner for this more general system, too.

We note the symmetry $h(-\check{\psi}) = h(\check{\psi})$ in Eq. (10) and in the more general equation above. This is a consequence of the aforementioned symmetry $X \mapsto -X$ for the complex Ginzburg–Landau equation. However, this evenness property holds only for the case when the basic periodic solution has $q = 0$. For general basic periodic solutions, as considered in Remark 2.2, this symmetry disappears. Instead, we have the relation $\tilde{h}(-\check{\psi}, -q) = \tilde{h}(\check{\psi}, q)$.

3. The approximation theorem for the $(\check{s}, \check{\psi})$-system

In this section we prove that solutions of the $(\check{s}, \check{\psi})$-system (7) can be approximated via the solutions of the conservation law (10).

Notations: For $\rho \geqslant 0$ and $m \in \mathbb{N}$, we define

$$L(\rho, m) = \{\hat{u} \in L^1(\mathbb{R}, \mathbb{C}) \,|\, \|\hat{u}\|_{L(\rho,m)} = \int |\hat{u}(k)| e^{\rho|k|}(1 + |k|^m)\, dk < \infty\}.$$

It is easy to see that for $\rho > 0$ the inverse Fourier transform $u = \mathscr{F}^{-1}\hat{u}$ is analytic in a complex strip $\mathscr{S}_\rho = \{z \in \mathbb{C} \,|\, |\mathrm{Im}\, z| < \rho\}$ with $\sup_{|\mathrm{Im}\, z| < \rho} |u(z)| \leqslant \|\hat{u}\|_{L(\rho,0)}$ (cf. [Kat76]). Then we define the Banach space

$$\mathscr{X}_\rho^m = \{u : \mathscr{S}_\rho \to \mathbb{C} \,|\, \hat{u} \in L(\rho, m), \|u\|_{\mathscr{X}_\rho^m} = \|\hat{u}\|_{L(\rho,m)} < \infty\}.$$

We have $||\hat{u} * \hat{v}||_{L(\rho,m)} \leqslant C||\hat{v}||_{L(\rho,m)}||\hat{u}||_{L(\rho,m)}$ for $\hat{u}, \hat{v} \in L(\rho, m)$ and since $uv = \mathscr{F}^{-1}(\hat{u} * \hat{v})$ consequently,

$$||uv||_{\mathscr{X}_\rho^m} \leqslant C||u||_{\mathscr{X}_\rho^m}||v||_{\mathscr{X}_\rho^m} \tag{13}$$

with a constant C independent of u and v. Thus, $\mathscr{X}_\rho^m$ is an algebra, i.e. closed under multiplication.

Theorem 3.1. *Fix the parameters α, $\beta \in \mathbb{R}$. For all $m \geqslant 1$, $\rho > 0$, and $\tau_0 > 0$ there exist $C_1 > 0$, $C_2 > 0$, $\tau_1 > 0$, and $\delta_0 > 0$ such that the following holds*:

Let $\psi^* \in C([0, \tau_0], \mathscr{X}_{2\rho}^0)$ be a solution of the conservation law (10) *with*

$$\sup_{\tau \in [0,\tau_0]} ||\psi^*(\tau)||_{\mathscr{X}_{2\rho}^0} \leqslant C_1$$

and let $s^* = s^*(\psi^*)$ be defined by the solution of (8). *Then for all $\delta \in (0, \delta_0)$ there exists a solution $(\check{s}, \check{\psi})$ of the Ginzburg–Landau equation* (7) *defined for $\tau \in [0, \tau_1]$ such that*

$$\sup_{\tau \in [0,\tau_1]} ||(\check{s}, \check{\psi})(\tau) - (s^*(\psi^*), \psi^*)(\tau)||_{\mathscr{X}^{m+3}_{\rho-\rho\tau/\tau_1} \times \mathscr{X}^{m+2}_{\rho-\rho\tau/\tau_1}} \leqslant C_2\delta.$$

In particular

$$|(\check{s}, \check{\psi})(\xi, \tau) - (s^*(\psi^*), \psi^*)(\xi, \tau)| \leqslant C_2\delta$$

for all $\tau \in [0, \tau_1]$ and $\xi \in \mathbb{R}$.

Remark 3.2. Since the error of order $\mathcal{O}(\delta)$ is small compared with the solutions (s^*, ψ^*) and $(\check{s}, \check{\psi})$ which are both of order $\mathcal{O}(1)$ for $\delta \to 0$, the dynamics of the conservation law (10) can be found in the $(\check{s}, \check{\psi})$-system (7), too.

At a first view it seems that our result is not of an optimal form since the approximation time τ_1 is smaller than the time τ_0 of the given solution—even when τ_0 is fixed smaller than the existence time τ_2 guaranteed by the Cauchy–Kowalevskaya theorem for the conservation law (10). But since the time τ_2 is independent of the fact that the time-periodic solutions are stable or unstable we do not expect any direct connection between τ_0 of the theorem, τ_1, and τ_2.

Remark 3.3. We refrain from greatest generality and work here with above definition of $\mathscr{X}_\rho^m$. As explained in [Ov76] the functional analytic setup used in [Ov76] also applies in other spaces. For example, our result is also true if $\mathscr{X}_\rho^m$ is replaced by the space of functions $u : \mathbb{R} \to \mathbb{R}$ for which $\sum_{n=0}^\infty \sum_{j=0}^m \rho^n \sup_{\xi \in \mathbb{R}} |\partial_\xi^{n+j} u(\xi)| < \infty$.

The remainder of this section is devoted to the proof of Theorem 3.1. Throughout, we assume that the parameters α and β are fixed. Moreover, possibly different constants are denoted with the same symbol C, if they can be chosen independent of $0 < \delta \ll 1$.

To start with, it is more convenient to rewrite system (7) in the abstract form

$$\partial_\tau \psi = A_\psi(\psi, s) + \partial_\xi B_\psi(\psi, s),$$

$$\partial_\tau s = A_s(\psi, s) + \delta^{-1} B_s(\psi, s), \tag{14}$$

where the linear terms A_ψ, A_s are given by

$$A_\psi(\psi, s) = \delta\partial_\xi^2\psi - 2\beta\partial_\xi s + \delta^2\alpha\partial_\xi^3 s, \qquad A_s(\psi, s) = \delta\partial_\xi^2 s - 2\delta^{-1}s - \alpha\partial_\xi\psi$$

and

$$B_\psi(\psi, s) = -\alpha\psi^2 - \beta s^2 + \delta^2\alpha(\partial_\xi^2 s)((1+s)^{-1} - 1) + 2\delta(\partial_\xi s)\psi(1+s)^{-1},$$

$$B_s(\psi, s) = -\psi^2 - \psi^2 s - 2\delta\alpha(\partial_\xi s)\psi - \delta\alpha(\partial_\xi\psi)s - 3s^2 - s^3.$$

(To simplify the notation, we have suppressed the dependence of $A_\psi, \dots, B_s$ on δ.) So instead of speaking of solutions $(\check{s}, \check{\psi})$ for (7), we speak of solutions (ψ, s) for (14). Define the residuals

$$\mathrm{Res}_\psi(\psi, s) = -\partial_\tau\psi + A_\psi(\psi, s) + \partial_\xi B_\psi(\psi, s),$$

$$\mathrm{Res}_s(\psi, s) = -\partial_\tau s + A_s(\psi, s) + \delta^{-1} B_s(\psi, s).$$

Obviously (ψ, s) is a solution of (14) if and only if $\mathrm{Res}_\psi(\psi, s) = \mathrm{Res}_s(\psi, s) = 0$.

Lemma 3.4. *Let ψ^* be a solution of the conservation law* (10) *and let $s^* = s^*(\psi^*)$ be defined by* (8). *For all $m, p \geq 0$, there exists $(\Psi^*, \mathbf{s}^*)$ and there exists a constant C_{Res} independent of $\delta \in [0, 1]$ such that*

$$\psi^* = \Psi^* + \mathcal{O}(\delta), \qquad s^* = \mathbf{s}^* + \mathcal{O}(\delta)$$

and

$$\sup_{\tau \in [0, \tau_0]} (\|\mathrm{Res}_\psi(\Psi^*, \mathbf{s}^*)(\tau)\|_{\mathscr{X}_\rho^{m+2}} + \|\mathrm{Res}_s(\Psi^*, \mathbf{s}^*)(\tau)\|_{\mathscr{X}_\rho^{m+3}}) \leq C_{\mathrm{Res}}\delta^p.$$

Notation: Here and in the following, all constants having to do with the residual which additionally can be chosen independent of δ are denoted with the same symbol C_{Res}.

Proof. Write

$$\Psi^* = \psi^* + \delta\psi_1^* + \delta^2\psi_2^* + \cdots + \delta^p\psi_p^*,$$

$$\mathbf{s}^* = s^* + \delta s_1^* + \delta^2 s_2^* + \cdots + \delta^p s_p^*.$$

We insert $(\Psi^*, \mathbf{s}^*)$ into (14) and equate coefficients of powers of δ. The δ^0 terms are precisely the equations satisfied by (ψ^*, s^*). At the δ^1 level, we obtain

$$
\begin{aligned}
0 = & -\partial_\tau \psi_1^* + \partial_\xi(-2\beta s_1^* - 2\alpha\psi^*\psi_1^* - 2\beta\psi^*\psi_1^*) + 2\partial_\xi((\partial_\xi s^*)\psi^*(1+s^*)^{-1}),\\
0 = & -\partial_\tau s^* - 2s_1^* - 2\psi_1^*\psi^* - 2\alpha(\partial_\xi s^*)\psi^* - 2\psi_1^*\psi^* s^* - (\psi^*)^2 s_1^*\\
& - \alpha\partial_\xi\psi^* - \alpha(\partial_\xi\psi^*)s^* - 6s_1^* s^* - 3(s^*)^2 s_1^*.
\end{aligned}
$$

This pair of linear nonhomogeneous equations can be solved for ψ_1^* and s_1^*. Equating higher powers of δ we solve inductively for $\psi_2^*, \dots, \psi_p^*; s_2^*, \dots, s_p^*$. $\square$

The main step is now to show that $(\Psi^*, \mathbf{s}^*)$ differs from a true solution $(\Psi, \mathbf{s})$ by a small error for sufficiently large p. We write a solution as approximation plus some error:

$$
\Psi = \Psi^* + \delta^p R_\psi, \qquad \mathbf{s} = \mathbf{s}^* + \delta^p R_s,
$$

where $R_\psi|_{\tau=0} = R_s|_{\tau=0} = 0$. The equations for the error are then given by

$$
\begin{aligned}
\partial_\tau R_\psi &= A_\psi(R_\psi, R_s) + \partial_\xi Q_\psi(R_\psi, R_s) + \delta^{-p}\,\mathrm{Res}_\psi(\Psi^*, \mathbf{s}^*),\\
\partial_\tau R_s &= A_s(R_\psi, R_s) + \delta^{-1} Q_s(R_\psi, R_s) + \delta^{-p}\,\mathrm{Res}_s(\Psi^*, \mathbf{s}^*),
\end{aligned}
\tag{15}
$$

where

$$
Q_\psi(R_\psi, R_s) = \delta^{-p}\{B_\psi(\Psi^* + \delta^p R_\psi, \mathbf{s}^* + \delta^p R_s) - B_\Psi(\Psi^*, s^*)\}
$$

and an analogous expression holds for Q_s.

Ultimately, we require that $p \geqslant 4$ and we prove that there exists a constant C independent of δ such that $\delta||R_\psi||_{\mathscr{X}^{m+2}_{\rho-\rho\tau/\tau_1}} \leqslant C$ and $\delta||R_s||_{\mathscr{X}^{m+3}_{\rho-\rho\tau/\tau_1}} \leqslant C$.

3.1. Smoothing

The Ginzburg–Landau equation in polar coordinates is quasi-linear, the lowest order linear terms do not possess any smoothing properties, and the smallness of the lowest order nonlinear terms is due to derivatives. Hence to obtain smoothing properties, the proof of the approximation property is made in the scale of Banach spaces $\mathscr{X}^m_\rho$ consisting of functions analytic in a strip of the complex plane which were defined above. The width ρ of the strip is made smaller with a linear rate as time evolves, i.e. $\rho(\tau) = \rho - C_\rho\tau$. The linear decay of ρ can be interpreted as an additional linear operator B in the equations for the error, defined by its symbol $\hat{B}(k) = -C_\rho|k|$ generating a linear semigroup $e^{\hat{B}(k)\tau}$.

Remark 3.5. Note that an artificial decay faster than $e^{-\rho|k|}$ in Fourier space is not possible due to the nonlinear terms. In a space with a decay faster than exponential, relation (13) no longer holds.

We take this semigroup as a basis for an operator $M(\tau)$ defined by its symbol $\hat{M}(\tau) = e^{(\rho - C_\rho \tau)|k|}$. Introduce the new variables

$$\mathscr{R}_\psi(\tau) = M(\tau) R_\psi(\tau) \quad \text{and} \quad \mathscr{R}_s(\tau) = M(\tau) R_s(\tau).$$

We have for instance that $\mathscr{R}_\psi(0) \in \mathscr{X}_0^m$ is equivalent to $R_\psi(0) \in \mathscr{X}_\rho^m$. We use the abbreviation $\mathscr{X}^m$ for $\mathscr{X}_0^m$ and denote the norm by $\|\ \|_m$.

Define

$$\mathscr{A}_\psi(\mathscr{R}_\psi, \mathscr{R}_s) = M(\tau) A_\psi(M(\tau)^{-1}\mathscr{R}_\psi, M(\tau)^{-1}\mathscr{R}_s),$$

$$\mathscr{Q}_\psi(\mathscr{R}_\psi, \mathscr{R}_s) = M(\tau) Q_\psi(M(\tau)^{-1}\mathscr{R}_\psi, M(\tau)^{-1}\mathscr{R}_s)$$

and similarly $\mathscr{A}_s$ and $\mathscr{Q}_s$. For $\mathscr{R} = (\mathscr{R}_\psi, \mathscr{R}_s)$ we obtain

$$\partial_\tau \mathscr{R} = \Lambda \mathscr{R} + \mathscr{N}(\mathscr{R}) + \mathrm{Res}, \tag{16}$$

where

$$\begin{aligned}
\Lambda \mathscr{R} &= (B\mathscr{R}_\psi + \mathscr{A}_\psi(\mathscr{R}_\psi, \mathscr{R}_s), B\mathscr{R}_s + \mathscr{A}_s(\mathscr{R}_\psi, \mathscr{R}_s)),\\
\mathscr{N}(\mathscr{R}) &= (\partial_\xi \mathscr{Q}_\psi(\mathscr{R}_\psi, \mathscr{R}_s), \delta^{-1}\mathscr{Q}_s(\mathscr{R}_\psi, \mathscr{R}_s)),\\
\mathrm{Res} &= (\delta^{-p} M(\tau)\mathrm{Res}_\psi, \delta^{-p} M(\tau)\mathrm{Res}_s).
\end{aligned}$$

Note that $\Lambda\mathscr{R}$ contains the autonomous linear terms, and $\mathscr{N}(\mathscr{R})$ contains both the nonautonomous linear terms and the nonlinear terms.

By construction we have

$$\sup_{\tau \in [0,\tau_0]} \|\delta^{-p} M(\tau)\mathrm{Res}_\psi\|_{m+2} \leqslant C_{\mathrm{Res}}, \qquad \sup_{\tau \in [0,\tau_0]} \|\delta^{-p} M(\tau)\mathrm{Res}_s\|_{m+3} \leqslant C_{\mathrm{Res}}.$$

Next we diagonalize the linear operator Λ. In Fourier space it is given by

$$\hat{\Lambda}(k) = (\mathscr{F}\Lambda\mathscr{F}^{-1})(k) = \begin{pmatrix} -\delta k^2 - C_\rho |k| & -i\alpha\delta^2 k^3 - 2i\beta k \\ -i\alpha k & -\delta k^2 - 2\delta^{-1} - C_\rho |k| \end{pmatrix}.$$

For given α, β we can choose $C_\rho > 0$ such that the eigenvalues $\lambda_j(k)$ of $\hat{\Lambda}(k)$ satisfy

$$\mathrm{Re}\,\lambda_1(k) \leqslant -C|k| - \delta k^2, \qquad \mathrm{Re}\,\lambda_2(k) \leqslant -\delta^{-1} - C|k| - \delta k^2$$

for some constant $C > 0$. The choice of C_ρ defines the possible approximation time τ_1. (When $\alpha\beta > -1$, we can take $C = C_\rho$ to be any positive number.)

Choose $P(k)$ so that

$$P^{-1}(k)\hat{\Lambda}(k)P(k) = \operatorname{diag}(\lambda_1(k), \lambda_2(k)).$$

(A specific choice of P is given in Lemma 3.6.) We introduce new coordinates $f = (f_1, f_2)$ by $\mathscr{R}(k) = P(k)f(k)$. The new variable f satisfies the equation

$$\partial_\tau f = \operatorname{diag}(\lambda_1, \lambda_2)f + \hat{\mathscr{N}}(f) + P^{-1}\,\mathrm{Res}, \tag{17}$$

where $\hat{\mathscr{N}}(f) = P^{-1}\mathscr{N}(Pf)$.

3.2. Estimates for the nonlinear terms

In this section, we estimate $||\hat{\mathscr{N}}(f)||_m$ for a specific choice of diagonalizing matrix P.

Lemma 3.6. *We can write* $P^{-1}\hat{\Lambda}P = \operatorname{diag}(\lambda_1, \lambda_2)$, *where*

$$P(k) = \begin{pmatrix} \mathcal{O}(\delta k) & \mathcal{O}(\delta k) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, \qquad P^{-1}(k) = \begin{pmatrix} \mathcal{O}((\delta k)^{-1}) & \mathcal{O}(1) \\ \mathcal{O}((\delta k)^{-1}) & \mathcal{O}(1) \end{pmatrix} \quad \text{as } |\delta k| \to \infty$$

and

$$P(k) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\delta k) \\ \mathcal{O}(\delta k) & \mathcal{O}(1) \end{pmatrix}, \qquad P^{-1}(k) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\delta k) \\ \mathcal{O}(\delta k) & \mathcal{O}(1) \end{pmatrix} \quad \text{as } |\delta k| \to 0.$$

The estimates are uniform in $\ell = \delta k$.

Proof. Define the matrices

$$\tilde{\Lambda}(k) = \begin{pmatrix} 0 & -i\alpha\delta^2 k^3 - 2i\beta k \\ -i\alpha k & -2\delta^{-1} \end{pmatrix}, \qquad \check{\Lambda}(\ell) = -\begin{pmatrix} 0 & i\alpha\ell^3 + 2i\beta\ell \\ i\alpha\ell & 2 \end{pmatrix}.$$

We have the relations

$$\tilde{\Lambda}(k) = \hat{\Lambda}(k) + (\delta k^2 + C_\rho |k|)I_2, \qquad \delta\tilde{\Lambda}(k) = \check{\Lambda}(\ell),$$

where $\ell = \delta k$. In particular, all three families of matrices are simultaneously diagonalized by the family of matrices P. Hence, we may focus on diagonalizing $\check{\Lambda}(\ell)$ which has the advantage of being independent of δ.

An explicit calculation shows that one possible choice of P is given by

$$P_0(\ell) = \begin{pmatrix} 1 + \sqrt{1 - \alpha^2\ell^4 - 2\alpha\beta\ell^2} & 1 - \sqrt{1 - \alpha^2\ell^4 - 2\alpha\beta\ell^2} \\ -i\alpha\ell & -i\alpha\ell \end{pmatrix}.$$

However, since eigenvectors are defined only up to scalar multiples, we may choose $P(\ell) = P_0(\ell)D(\ell)$ where D is any diagonal matrix.

For $|\ell| \geqslant 1$, we take $D(\ell) = (1/\ell)I_2$ so that

$$P = \begin{pmatrix} \mathcal{O}(\ell) & \mathcal{O}(\ell) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} \mathcal{O}(\ell^{-1}) & \mathcal{O}(1) \\ \mathcal{O}(\ell^{-1}) & \mathcal{O}(1) \end{pmatrix} \quad \text{as } |\ell| \to \infty.$$

For $|\ell| \leqslant 1$, we take $D(\ell) = \mathrm{diag}(1, 1/\ell)$ so that

$$P = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\ell) \\ \mathcal{O}(\ell) & \mathcal{O}(1) \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\ell) \\ \mathcal{O}(\ell) & \mathcal{O}(1) \end{pmatrix} \quad \text{as } |\ell| \to 0. \qquad \square$$

We will use this lemma in the following form.

Proposition 3.7. (a) *Writing $\mathcal{R} = Pf$, we have*

$$\|\mathcal{R}_\psi\|_m \leqslant C(\|f\|_m + \delta\|f\|_{m+1}), \qquad \|\mathcal{R}_s\|_m \leqslant C\|f\|_m.$$

(b) *For the inverse $P^{-1} = (q_{ij})$ we have*

$$|q_{11}(k)| \leqslant C\min(1, (\delta k)^{-1}), \qquad |q_{12}(k)| \leqslant C\min(1, \delta k),$$

$$|q_{21}(k)| \leqslant C\min(\delta k, (\delta k)^{-1}), \qquad |q_{22}(k)| \leqslant C.$$

Remark 3.8. It follows from Lemma 3.6 that in order to prove Theorem 3.1, it suffices to show that $\delta\|f\|_{m+3} \leqslant C$.

Lemma 3.9. *Write $\mathcal{R} = Pf$ and $\hat{\mathcal{N}}(f) = P^{-1}\mathcal{N}(Pf)$ for P specified in Lemma* 3.6. *Let $\hat{\mathcal{N}} = (\hat{\mathcal{N}}_1, \hat{\mathcal{N}}_2)$. Then there exists $C_1 > 0$ with $C_1 \to 0$ as $\|\psi^*\|_{\mathcal{X}^0_{2\rho}} \to 0$, and there exists $\delta_0, C > 0$ such that*

$$\|\hat{\mathcal{N}}_1(f)\|_m \leqslant C_1(\|f\|_{m+1} + \delta\|f\|_{m+2}) + C\delta^{p-1}\|f\|^2_{m+2}, \tag{18}$$

$$\|\hat{\mathcal{N}}_2(f)\|_m \leqslant C_1(\delta^{-1}\|f\|_m + \|f\|_{m+1} + \delta\|f\|_{m+2}) + C\delta^{p-1}\|f\|^2_{m+2} \tag{19}$$

for all $\delta \in (0, \delta_0)$ and all $\mathcal{R} = (\mathcal{R}_\psi, \mathcal{R}_s)$ satisfying $\|\delta^p\mathcal{R}_\psi\|_{m+1} + \|\delta^p\mathcal{R}_s\|_{m+2} \leqslant M$ for some constant $M > 0$ independent of δ.

In the remainder of this section, we verify the estimates in Lemma 3.9.

Recall that $\mathcal{N} = (\partial_\xi \mathcal{Q}_\psi, \delta^{-1}\mathcal{Q}_s)$ consists of both linear nonautonomous terms as well as nonlinear terms. Accordingly, we write

$$Q_\psi = L_\psi + \delta^p N_\psi, \qquad Q_s = L_s + \delta^p N_s,$$

where L_ψ, L_s are the linear nonautonomous terms and N_ψ, N_s are the nonlinear terms.

Lemma 3.10. *There exists a $C_1 > 0$ such that for all $\mathcal{R}_\psi$ and $\mathcal{R}_s$ and all $\delta \in (0, 1]$*

$$||L_\psi(\mathcal{R}_\psi, \mathcal{R}_s)||_m \leqslant C_1(||\mathcal{R}_\psi||_m + ||\mathcal{R}_s||_m + \delta||\mathcal{R}_s||_{m+1} + \delta^2||\mathcal{R}_s||_{m+2}),$$

$$||L_s(\mathcal{R}_\psi, \mathcal{R}_s)||_m \leqslant C_1(||\mathcal{R}_\psi||_m + ||\mathcal{R}_s||_m + \delta||\mathcal{R}_\psi||_{m+1} + \delta||\mathcal{R}_s||_{m+1}).$$

The constant C_1 can be chosen to satisfy $C_1 \to 0$ for $||\psi^*||_{\mathcal{X}^0_{2\rho}} \to 0$.

For all $M > 0$ there exist $\delta_0, C > 0$ such that for all $\delta \in (0, \delta_0)$ and all $\rho' \in (0, \rho)$ and $\mathcal{R}_\psi$ and $\mathcal{R}_s$ with $||\delta^p \mathcal{R}_\psi||_{m+1} + ||\delta^p \mathcal{R}_s||_{m+2} \leqslant M$ we have

$$||N_\psi(\mathcal{R}_\psi, \mathcal{R}_s)||_m \leqslant C(||\mathcal{R}_\psi||^2_m + ||\mathcal{R}_s||^2_{m+2}),$$

$$||N_s(\mathcal{R}_\psi, \mathcal{R}_s)||_m \leqslant C(||\mathcal{R}_\psi||^2_{m+1} + ||\mathcal{R}_s||^2_{m+1}).$$

Proof. An explicit calculation shows that

$$L_\psi(R_\psi, R_s) = -2\alpha\Psi^* R_\psi - 2\beta \mathbf{s}^* R_s + K(R_\psi, R_s),$$

$$\begin{aligned} K(R_\psi, R_s) = {}& 2\delta\left(\frac{(\partial_\xi \mathbf{s}^*)R_\psi}{1+\mathbf{s}^*} + \frac{(\partial_\xi R_s)\Psi^*}{1+\mathbf{s}^*} - \frac{(\partial_\xi \mathbf{s}^*)\Psi^*}{(1+\mathbf{s}^*)^2} R_s - \alpha\delta \frac{\partial_\xi^2 \mathbf{s}^*}{(1+\mathbf{s}^*)^2} R_s\right) \\ & + \alpha\delta^2\left(\frac{1}{1+\mathbf{s}^*} - 1\right)\partial_\xi^2 R_s, \end{aligned}$$

$$\begin{aligned} N_\psi(R_\psi, R_s) = {}& -\alpha R_\psi^2 - \beta R_s^2 + \delta^{-2p}\left(\frac{\alpha\delta^2\partial_\xi^2(\mathbf{s}^* + \delta^p R_s)}{1 + (\mathbf{s}^* + \delta^p R_s)} - \frac{2\delta(\partial_\xi(\mathbf{s}^* + \delta^p R_s))(\Psi^* + \delta^p R_\psi)}{1 + (\mathbf{s}^* + \delta^p R_s)}\right) \\ & - \delta^{-p+2}\alpha\partial_\xi^2 R_s - \delta^{-p} K(R_\psi, R_s) - \delta^{-2p}\left(\frac{\alpha\delta^2\partial_\xi^2 \mathbf{s}^*}{1+\mathbf{s}^*} - \frac{2\delta(\partial_\xi \mathbf{s}^*)\Psi^*}{1+\mathbf{s}^*}\right), \end{aligned}$$

$$\begin{aligned} L_s(R_\psi, R_s) = {}& -2\Psi^* R_\psi - 2\Psi^* \mathbf{s}^* R_\psi - (\Psi^*)^2 R_s - 6s^* R_s - 3(\mathbf{s}^*)^2 R_s - 2\alpha\delta(\partial_\xi R_s)\Psi^* \\ & - \alpha\delta(\partial_\xi \Psi^*)R_s - 2\alpha\delta(\partial_\xi \mathbf{s}^*)R_\psi - \alpha\delta(\partial_\xi R_\psi)\mathbf{s}^*, \end{aligned}$$

$$\begin{aligned} N_s(R_\psi, R_s) = {}& -R_\psi^2 - 2\Psi^* R_\psi R_s - R_\psi^2 \mathbf{s}^* - \delta^p R_\psi^2 R_s - 2\alpha\delta(\partial_\xi R_s)R_\psi - \alpha\delta(\partial_\xi R_\psi)R_s \\ & - 3R_s^2 - 3\mathbf{s}^* R_s^2 - \delta^p R_s^3. \end{aligned}$$

The result follows easily. $\square$

It follows from Proposition 3.7(a) and Lemma 3.10 that

$$||L_\psi(\mathscr{R}_\psi,\mathscr{R}_s)||_m \leqslant C_1(||f||_m + \delta||f||_{m+1} + \delta^2||f||_{m+2}),$$

$$||L_s(\mathscr{R}_\psi,\mathscr{R}_s)||_m \leqslant C_1(||f||_m + \delta||f||_{m+1} + \delta^2||f||_{m+2}),$$

$$||N_\psi(\mathscr{R}_\psi,\mathscr{R}_s)||_m \leqslant C||f||^2_{m+2},$$

$$||N_s(\mathscr{R}_\psi,\mathscr{R}_s)||_m \leqslant C||f||^2_{m+2}.$$

Recall that

$$\mathscr{N} = (\partial_\xi(L_\psi + \delta^p N_\psi), \delta^{-1}(L_s + \delta^p N_s))$$

and so

$$||\mathscr{N}_1(Pf)||_m \leqslant C_1(||f||_{m+1} + \delta||f||_{m+2} + \delta^2||f||_{m+3}) + C\delta^p||f||^2_{m+3},$$

$$||\mathscr{N}_2(Pf)||_m \leqslant C_1(\delta^{-1}||f||_m + ||f||_{m+1} + \delta||f||_{m+2}) + C\delta^{p-1}||f||^2_{m+2}.$$

Applying P^{-1} and using Proposition 3.7(b) completes the proof of Lemma 3.9. □

Remark 3.11. By a more detailed consideration it is easy to see that $\hat{\mathscr{N}}_1(f)$ and $\hat{\mathscr{N}}_2(f)$ are sums of finitely many terms, where for the individual estimate for each of these terms only one of the terms on the right-hand side of (18) and (19) is necessary. This remark also applies in a similar fashion to Proposition 3.7(a).

3.3. *Optimal regularity argument*

Recall that Eq. (17) reads: $\partial_\tau f = \mathrm{diag}(\lambda_1,\lambda_2)f + \hat{\mathscr{N}}(Pf) + P^{-1}\,\mathrm{Res}$. The semigroups associated to the eigenvalues $\lambda_1(k)$ and $\lambda_2(k)$ satisfy

$$\begin{aligned}\sup_{k\in\mathbb{R}} |e^{\lambda_1(k)\tau}k^n| &\leqslant \sup_{k\in\mathbb{R}} |e^{(-C|k|-\delta k^2)\tau}k^n| \leqslant \sup_{k\in\mathbb{R}} |e^{-C|k|\tau}k^{n-\tilde{n}}| \sup_{k\in\mathbb{R}} |e^{-\delta k^2\tau}k^{\tilde{n}}| \\ &\leqslant \sup_{s\in\mathbb{R}} |e^{-C|s|}(s/\tau)^{n-\tilde{n}}| \sup_{s\in\mathbb{R}} |e^{-s^2}(s/\sqrt{\delta\tau})^{\tilde{n}}| \leqslant C\tau^{\tilde{n}-n}(\tau\delta)^{-\tilde{n}/2}\end{aligned}$$

and

$$\sup_{k\in\mathbb{R}} |e^{\lambda_2(k)\tau}k^n| \leqslant \sup_{k\in\mathbb{R}} |e^{(-\delta^{-1}-C|k|-\delta k^2)\tau}k^n| \leqslant Ce^{-\delta^{-1}\tau}\tau^{\tilde{n}-n}(\tau\delta)^{-\tilde{n}/2}$$

for $0\leqslant\tilde{n}\leqslant n$. Hence we have

$$||e^{\lambda_1\tau}u||_{m+r} \leqslant C\tau^{\tilde{r}-r}(\tau\delta)^{-\tilde{r}/2}||u||_m, \tag{20}$$

$$||e^{\lambda_2 \tau} u||_{m+r} \leqslant C e^{-\delta^{-1}\tau} \tau^{\tilde{r}-r} (\tau\delta)^{-\tilde{r}/2} ||u||_m \tag{21}$$

for $0 \leqslant \tilde{r} \leqslant r$.

Since the nonlinear terms contain as many derivatives as the linear ones we need an optimal regularity result. We consider Hölder functions $g : [0, \tau] \to \mathscr{X}^m$ with Hölder exponent $\theta \in (0,1)$ and define

$$||g||_{\theta,m,\tau} = ||g||_{\theta,m} = ||g||_{C^\theta([0,\tau],\mathscr{X}^m)}.$$

If $g = (g_1, g_2)$, we define $||g||_{\theta,m} = ||g_1||_{\theta,m} + ||g_2||_{\theta,m}$.

Lemma 3.12. *Suppose that $g_j|_{\tau=0} = 0$ for $j = 1, 2$. For $r \in [0, 2]$ the solutions f_j of*

$$\partial_\tau f_j = \lambda_j f_j + g_j, \quad j = 1, 2 \tag{22}$$

satisfy

$$||f_1||_{\theta,m+r} \leqslant C\delta^{\min(0,1-r)} ||g_1||_{\theta,m} \quad \text{and} \quad ||f_2||_{\theta,m+r} \leqslant C\delta^{1-r} ||g_2||_{\theta,m}.$$

Proof. The proof follows by direct calculation based on a classical optimal regularity result (cf. [Am95]) using the estimates on the linear semigroup $e^{\mathscr{D}\tau} = \mathrm{diag}(e^{\lambda_1\tau}, e^{\lambda_2\tau})$ from above. The details are as follows. Using (20), we estimate

$$\begin{aligned}
&\left|\left|\int_0^\tau e^{\lambda_1(k)(\tau-\tau')} g_1(\tau')\, d\tau'\right|\right|_{m+r} \\
&\quad \leqslant \left|\left|\int_0^\tau e^{\lambda_1(k)(\tau-\tau')} (g_1(\tau') - g_1(\tau))\, d\tau'\right|\right|_{m+r} + \left|\left|\int_0^\tau e^{\lambda_1(k)(\tau-\tau')}\, d\tau' g_1(\tau)\right|\right|_{m+r} \\
&\quad \leqslant C\int_0^\tau \tau'^{\tilde{r}-r} (\tau'\delta)^{-\tilde{r}/2} \tau'^\theta\, d\tau' ||g_1||_{\theta,m} + \left|\left|\frac{1-e^{\lambda_1(k)\tau}}{\lambda_1(k)} g_1(\tau)\right|\right|_{m+r} \\
&\quad = s_1 + s_2.
\end{aligned}$$

In order to estimate s_2 we compute

$$\begin{aligned}
s_2 &\leqslant \sup_{k\in\mathbb{R}} \left|\frac{1-e^{\lambda_1(k)\tau}}{\lambda_1(k)}(1+|k|^r)\right| \tau^\theta ||g_1||_{\theta,m} \leqslant C\left(1 + \sup_{|k|\geqslant 1} \frac{|k|^r}{|k| + \delta|k|^2}\right)\tau^\theta ||g_1||_{\theta,m} \\
&\leqslant C\left(1 + \sup_{|k|\geqslant 1} \frac{|k|^{r-1}}{1+\delta|k|}\right)\tau^\theta ||g_1||_{\theta,m} \leqslant C\delta^{\min(0,1-r)} \tau^\theta ||g_1||_{\theta,m}.
\end{aligned}$$

The first term s_1 is estimated by

$$s_1 \leq C\delta^{-\tilde{r}/2} \frac{1}{\tilde{r}/2 - r + \theta + 1} \tau^{\tilde{r}/2-r+\theta+1} ||g_1||_{\theta,m} \leqslant C\delta^{\min(0,1-r)} \tau^\theta ||g_1||_{\theta,m},$$

where we have chosen $\tilde{r}/2 = r-1$ for $r \in [1,2]$ and $\tilde{r}/2 = 0$ for $r \in [0,1]$. (Since the sum cannot be estimated better than s_2 we have not optimized the last estimate in terms of δ.)

In a similar way we obtain

$$\left|\left|\int_0^\tau e^{\lambda_2(k)(\tau-\tau')} g_2(\tau')\, d\tau'\right|\right|_{m+r}$$
$$\leqslant C\int_0^\tau e^{-\delta^{-1}\tau'}\tau'^{\tilde{r}-r}(\tau'\delta)^{-\tilde{r}/2}\tau'^{\theta}\, d\tau' ||g_2||_{\theta,m} + \left|\left|\frac{1-e^{\lambda_2(k)\tau}}{\lambda_2(k)} g_2(\tau)\right|\right|_{m+r}$$
$$\leqslant C\delta^{1-r}\tau^{\theta}||g_2||_{\theta,m},$$

where we used the following two variants of estimates:

(a) For $r \in [1,2]$ we estimate $e^{-\delta^{-1}\tau'} \leqslant 1$ and the integral by $C\tau^{\tilde{r}/2-r+1+\theta}\delta^{-\tilde{r}/2}$. As above, we choose $\tilde{r}/2 - r + 1 = 0$ which gives δ^{1-r}.

(b) For $r \in [0,1]$ we introduce $\delta^{-1}\tau' = \tilde{\tau}$ and estimate the integral by

$$C\int_0^\infty e^{-\tilde{\tau}}(\delta\tilde{\tau})^{\tilde{r}/2-r+\theta}\delta^{-\tilde{r}/2}\delta\, d\tilde{\tau}$$

which gives again δ^{1-r}.

These two estimates additionally show the Hölder-continuity with exponent θ for $\tau = 0$. In a very similar fashion, the Hölder-continuity for $\tau > 0$ is obtained with the same estimates in terms of δ (cf. [Am95]). $\square$

Proposition 3.13.

$$||f||_{\theta,m+2,\tau} \leq C_4||f||_{\theta,m+2,\tau} + C_5\delta^{p-2}||f||^2_{\theta,m+2,\tau} + C_{\mathrm{Res}}, \tag{23}$$

where the constants C_4 and C_5 are independent of δ, and $C_4 \to 0$ for $||\psi^*||_{\mathscr{X}^0_{2\rho}} \to 0$.

Proof. We apply Lemma 3.12 to estimate each term in (17) successively using variation of constants. Note that $\mathscr{R}_\psi = \mathscr{R}_s = 0$ at $\tau = 0$, and so $L_\psi = L_s = N_\psi = N_s = 0$ at $\tau = 0$, as required to apply Lemma 3.12. By Lemma 3.9 and Remark 3.11, the nonlinearity $\hat{\mathscr{N}}$ consists of terms satisfying estimates in the $||\ ||_m$ norm of the form

$$\delta^{-1}||f||_m, \qquad ||f||_{m+1}, \qquad \delta||f||_{m+2}, \qquad \delta^{p-1}||f||^2_{m+2}.$$

Moreover, the terms estimated by $\delta^{-1}||f||_m$ appear only in the second component $\hat{\mathscr{N}}_2$. Such terms are handled by taking $r = 0$ in Lemma 3.12. The remaining terms are handled by taking $r = 1, 2$ in Lemma 3.12 as appropriate. In this way, we obtain

$$||f||_{\theta,m+2} \leqslant C_4||f||_{\theta,m+2} + C_5\delta^{p-2}||f||^2_{\theta,m+2} + ||\mathrm{Res}||_{\theta,m+2}.$$

Since the approximation ψ^* is arbitrarily smooth compared with the error, $||\mathrm{Res}||_{\theta,m+2} \leqslant C_{\mathrm{Res}}$. $\square$

Proposition 3.14. *There exists a constant $C_6 > 0$ (independent of δ) such that*

$$\lim_{\tau\to 0} ||f||_{\theta,m+2,\tau} \leqslant C_6\delta^{-1}.$$

Proof. We establish this estimate for the C^1 norm. By Lemma 3.6, $||f||_{1,m+2,\tau} \leqslant \delta^{-1}||R||_{1,m+2,\tau}$ where by definition $R_\psi = \delta^{-p}(\Psi - \Psi^*)$, $R_s = \delta^{-p}(s - s^*)$. At the C^0 level, we have $f(0) = 0$ so it suffices to consider the contribution from the first derivative. We estimate directly that $|\partial_t(\Psi - \Psi^*)| \leqslant C\delta^p$ and similarly for $s - s^*$. $\square$

We now complete the proof of Theorem 3.1. To simplify the exposition, we do not strive for optimal choices of the constants C_1 and so on in the statement of the theorem. Rewrite (23) as

$$||f||_{\theta,m+2,\tau}(1 - C_4 - C_5\delta^{p-2}||f||_{\theta,m+2,\tau}) \leqslant C_{\mathrm{Res}}. \tag{24}$$

Choose C_1 sufficiently small that $C_4 < \frac{1}{4}$ and set

$$\delta_0 = \min\left\{1, \frac{1}{4C_{\mathrm{Res}}}, \frac{1}{4C_5(1 + C_6)}\right\}.$$

For fixed $\delta \in (0, \delta_0)$, let $\tau^*(\delta)$ denote the maximum value of $\tau \in [0, \tau_1]$ such that $||f||_{\theta,m+2,\tau} \leqslant (1 + C_6)\delta^{-1}$. (This is well defined by Proposition 3.14.) Then

$$C_5\delta^{p-2}||f||_{\theta,m+2,\tau} \leqslant C_5(1 + C_6)\delta^{p-3} \leqslant C_5(1 + C_6)\delta_0 \leqslant \tfrac{1}{4},$$

where we assume that $p \geqslant 4$. Hence $(1 - C_4 - C_5\delta^{p-2}||f||_{\theta,m+2,\tau}) \geqslant \frac{1}{2}$. By (24),

$$||f||_{\theta,m+2,\tau} \leqslant 2C_{\mathrm{Res}} \leqslant \tfrac{1}{2}\delta_0^{-1} \leqslant \tfrac{1}{2}\delta^{-1} < (1 + C_6)\delta^{-1}.$$

This would contradict the maximality of $\tau^*(\delta)$ so we conclude that $\tau^*(\delta) = \tau_1$ independent of δ. We have shown that $\delta||f||_{\theta,m+2} \leqslant (1 + C_6)$ on $[0, \tau_1]$. By Remark 3.8, this completes the proof of Theorem 3.1. $\square$

Remark 3.15. An alternative approach [Me98,Me99] to justify the conservation law (10) for the Ginzburg–Landau equation (6) is to consider spaces of functions $s(X,T)$, $\psi(X,T)$ that lie in the Banach space $\mathscr{X}$ of Fourier transforms of Borel measures with bounded total variation norm. We briefly describe the results of this approach, referring to [Me98,Me99] for details.

The starting point is the (s, ψ) system (6) or more generally system (12) obtained by including higher order terms in the complex Ginzburg–Landau equation. It can be shown that locally in Banach space there is a one-to-one correspondence between "essential solutions" for (12) and essential solutions for a pseudo-differential (in time and space) equation of the form

$$\partial_T \psi = \partial_X H(\psi) = \partial_X \{(\alpha\beta + 1)\partial_X \psi + (\beta - \alpha)\psi^2 + \cdots\},$$

where $\partial_X H$ is a constant coefficient pseudo-differential operator that respects the symmetry $(X \to -X, \psi \to -\psi)$. The scaling $\tau = \delta^2 T$, $\xi = \delta X$, $\psi = \delta\hat{\psi}$ leads to the Burgers equation $\partial_\tau \hat{\psi} = (\alpha\beta + 1)\partial_\xi^2 \hat{\psi} + (\beta - \alpha)\partial_\xi \hat{\psi} + \mathcal{O}(\delta^2)$ whereas the scaling $\tau = \delta T$, $\xi = \delta X$ leads to the conservation law $\partial_\tau \psi = \partial_\xi h(\psi) + \mathcal{O}(\delta)$ in which we are interested in this paper.

4. The approximation theorem for the complex Ginzburg–Landau equation

In this section, we transfer the approximation result of Theorem 3.1, i.e. that the $(\check{s}, \check{\psi})$-system (14) can be approximated via solutions of the conservation law (10), back to the complex Ginzburg–Landau equation (3). It turns out that we cannot expect validity uniformly for all $X \in \mathbb{R}$, but validity only uniformly for all $X \in I_\delta$ with I_δ an interval of length $\mathcal{O}(\delta^{-r})$ with arbitrary but fixed $r > 0$, depending on the chosen rate of approximation. Moreover, we have to allow for a global phase $\exp(i\phi(0, T))$.

Our starting point is the relation

$$A(X, T) = (1 + \check{s}(\delta X, \delta T))\exp\left(i \int_0^X \check{\psi}(\delta X', \delta T)\, dX' + i\omega_0 T + i\phi(0, T)\right)$$

which defines the solution A of the complex Ginzburg–Landau equation (3) in terms of solutions $(\check{s}, \check{\psi})$ of (14). These solutions are approximated by

$$A_{\text{app}}(X, T) = (1 + \mathbf{s}^*(\delta X, \delta T))\exp\left(i \int_0^X \Psi^*(\delta X', \delta T)\, dX' + i\omega_0 T\right),$$

where we have to use the improved approximations $(\mathbf{s}^*, \Psi^*)$ constructed in the proof of Theorem 3.1 from the solution ψ^* of the conservation law (10). Then we obtain

$$\begin{aligned} &|\exp(-i\phi(0, T)) \quad A(X, T) - A_{\text{app}}(X, T)| \\ &\leqslant \left|(1 + \check{s}(\delta X, \delta T))\exp\left(i \int_0^X \check{\psi}(\delta X', \delta T)\, dX' + i\omega_0 T\right)\right. \end{aligned}$$

$$
\begin{aligned}
&- (1+\mathbf{s}^*(\delta X,\delta T))\exp\left(i\int_0^X \Psi^*(\delta X',\delta T)\,dX' + i\omega_0 T\right)\Bigg| \\
\leqslant & \left|(1+\check{s}(\delta X,\delta T))\exp\left(i\int_0^X \check{\psi}(\delta X',\delta T)\,dX' + i\omega_0 T\right)\right. \\
&\left. - (1+\check{s}(\delta X,\delta T))\exp\left(i\int_0^X \Psi^*(\delta X',\delta T)\,dX' + i\omega_0 T\right)\right| \\
&+ \left|(1+\check{s}(\delta X,\delta T))\exp\left(i\int_0^X \Psi^*(\delta X',\delta T)\,dX' + i\omega_0 T\right)\right. \\
&\left. - (1+\mathbf{s}^*(\delta X,\delta T))\exp\left(i\int_0^X \Psi^*(\delta X',\delta T)\,dX' + i\omega_0 T\right)\right| \\
\leqslant & |1+\check{s}(\delta X,\delta T))|\left|\exp\left(i\int_0^X \check{\psi}(\delta X',\delta T)\,dX'\right)\right. \\
&\left. - \exp\left(i\int_0^X \Psi^*(\delta X',\delta T)\,dX'\right)\right| + |\mathbf{s}^*(\delta X,\delta T)) - \check{s}(\delta X,\delta T))| \\
\leqslant & C\int_0^X |\check{\psi}(\delta X',\delta T) - \Psi^*(\delta X',\delta T)|\,dX' + C\delta^p \\
\leqslant & \int_0^X C\delta^p dX' + C\delta^p \leqslant C\delta^p(1+|X|)
\end{aligned}
$$

using the approximation result of Theorem 3.1. Thus, we have proved

Theorem 4.1. *For all $m\geqslant 1$, $p\in\mathbb{N}$, $\tau_0>0$, and $\rho>0$ there exist $C_1>0$, $C_2>0$, $\tau_1>0$, and $\delta_0>0$ such that for all $\delta\in(0,\delta_0)$ the following holds. Let $\psi^*\in C([0,\tau_0],\mathscr{X}^0_{2\rho})$ be a solution of the conservation law* (10) *associated to the complex Ginzburg–Landau equation* (3) *with*

$$
\sup_{\tau\in[0,\tau_0]} ||\psi^*(\tau)||_{\mathscr{X}^0_{2\rho}} \leqslant C_1
$$

and let $(\mathbf{s}^*,\Psi^*)$ be the improved approximation constructed in the proof of Theorem 3.1 *with approximation rate $\mathcal{O}(\delta^p)$. Then there exist solutions of the complex Ginzburg–Landau equation* (3) *such that for all $r\in(0,p)$ we have*

$$
\begin{aligned}
\sup_{T\in[0,\tau_1/\delta]}\ \sup_{|X|\leqslant\delta^{-r}} &\Bigg| A(X,T)\exp(-i\phi(0,T)) - (1+\mathbf{s}^*(\psi^*)(\delta X,\delta T)) \\
&\exp\left(i\int_0^X \Psi^*(\delta X',\delta T)\,\mathrm{d}X' + i\omega_0 T\right)\Bigg| \leqslant C_2\delta^{p-r}.
\end{aligned}
$$

Hence the approximation result holds uniformly on intervals larger than the natural spatial scale $(r = 1)$ of the conservation law. Due to the translation invariance of the original system this holds for all intervals of length $\mathcal{O}(\delta^{-r})$ with $r > 0$ arbitrary, but fixed, using redefined approximations. Note that we have to allow a global x-independent phase $\exp(-i\phi(0, T))$ where for $\phi(0, T)$ there is no suitable estimate on the long $\mathcal{O}(1/\delta)$-time interval.

By taking $\check{\psi} = \delta^p$ and $\psi^* = 0$ we have to compare $e^{i\delta^p X}$ with 1 which shows that estimates uniformly valid for all $X \in \mathbb{R}$ cannot be expected. A uniform estimate can only be expected for $\check{\psi}$ and ψ^* spatially localized.

5. Application: the weakly unstable Taylor–Couette problem

In the remainder of the paper, we explain how the dynamics of the conservation law can also be found in classical pattern forming systems. As an example of such a system we consider the weakly unstable Taylor–Couette problem. The proof is based on the fact that the Taylor–Couette problem close to the first instability can be approximated by the Ginzburg–Landau equation.

The Taylor–Couette problem consists of finding the velocity field for a viscous incompressible fluid filling the domain $\Omega = \mathbb{R} \times \Sigma$ between two concentric rotating infinite cylinders, where $\Sigma \subset \mathbb{R}^2$ denotes the bounded cross-section. The flow in between the rotating cylinders is described by the Navier–Stokes equations on Ω with no-slip boundary conditions. We denote the inner and outer radii of the cylinders by R_1 and R_2, and the angular velocities of the inner and outer cylinders by ω_1 and ω_2. In cylindrical coordinates (x, r, Θ), the cross-section Σ is defined by $R_1 < r < R_2$ and $\Theta \in S_{2\pi} = \mathbb{R}/2\pi\mathbb{Z}$. The cartesian coordinates in the bounded cross-section are denoted with $z = (z_1, z_2) \subset \Sigma \subset \mathbb{R}^2$. We have the nondimensionalized parameters

$$\omega = \omega_2/\omega_1, \qquad \eta = R_1/R_2, \qquad \mathcal{R} = R_1\omega_1 d/v,$$

where $d = R_2 - R_1$, v is the kinetic viscosity, and $\mathcal{R}$ is called the Reynolds number.

This physical system possesses a steady-state solution, called Couette flow, having a purely azimuthal form (streamlines are concentric circles). For small Reynolds number $\mathcal{R}$, this solution is asymptotically stable with some exponential rate. The deviation (U, p) from the Couette flow U_{Cou} satisfies the Navier–Stokes equations

$$\begin{aligned} \partial_t U &= \Delta U - \mathcal{R}[(U_{\mathrm{Cou}} \cdot \nabla)U + (U \cdot \nabla)U_{\mathrm{Cou}} + (U \cdot \nabla)U] - \nabla p, \\ \nabla \cdot U &= 0 \end{aligned} \tag{25}$$

with boundary conditions $U = 0$ at $r = \eta/(1 - \eta)$ and $r = 1/(1 - \eta)$. In order to solve this problem uniquely for the velocity U and pressure gradient ∇p we add the flux condition $[U_{(x)}]_\Sigma = \frac{1}{|\Sigma|}\int_{z \in \Sigma} U_{(x)}(x, z)\, dz = 0$, where $U_{(x)}$ stands for the velocity component along the x-axis. We refer to [CI94] for more details.

The trivial branch of solutions, the Couette flow, $U \equiv 0$ in (25), becomes unstable if the Reynolds number $\mathscr{R}$ goes beyond a certain threshold of instability $\mathscr{R}_c$. Due to the translation invariance of (25) the linearized system possesses solutions $e^{ikx}\varphi_n(k,y)e^{\lambda_n(k)t}$ with $k \in \mathbb{R}$, $n \in \mathbb{N}$ and $\varphi_n(k,y) \in \mathbb{C}^3$. Without loss of generality we assume $\mathrm{Re}\,\lambda_n \geqslant \mathrm{Re}\,\lambda_{n+1}$ for all $n \in \mathbb{N}$.

For $\eta = R_1/R_2$ close to 1 there exists an ω_b, such that for $\omega > \omega_b$ at $\mathscr{R} = \mathscr{R}_c$ the real-valued curve $k \mapsto \lambda_1(k)$ touches the imaginary axis and that for $\omega < \omega_b$ the two complex conjugate curves $k \mapsto \lambda_1(k)$ and $k \mapsto \lambda_2(k)$ with $\lambda_2(k) = \overline{\lambda_1(k)}$ touch the imaginary axis at some wave number $k = k_c \neq 0$. In both cases all other curves are strictly bounded away from the imaginary axis. The first case is called PRI and the second case PRII in the following. (These bifurcations are often referred to as steady-state bifurcation with nonzero critical wavenumber and Hopf bifurcation with nonzero critical wave number [Me00].)

In the parameter region PRI, the Taylor–Couette problem can be approximated by the real Ginzburg–Landau equation which can in turn be approximated by a phase diffusion equation [MS03]. We concentrate on the parameter region PRII, where the Taylor–Couette problem can be approximated by a system of two coupled complex Ginzburg–Landau equations for amplitudes A_1, A_2 corresponding to the curve of eigenvalues λ_1, λ_2. These equations decouple for $A_2 \equiv 0$ and also for $A_1 \equiv A_2$. Thus this problem possesses two distinct families of solutions which can be described by a single complex Ginzburg–Landau equation (cf. [Schn99]). These families are modulations of axially spatially periodic traveling wave and standing wave solutions whose existence can be deduced by the implicit function theorem ("Hopf bifurcation with $\mathbf{O}(2)$ symmetry" [CI94,GSS88]). The complex Ginzburg–Landau equations can now each be approximated by a conservation law of the form

$$\partial_\tau \psi = \partial_\xi h(\psi) \tag{26}$$

(for two different functions h).

To be more precise, we introduce the small bifurcation parameter $\varepsilon^2 = \mathscr{R} - \mathscr{R}_c$. The ansatz

$$U = \varepsilon A(\varepsilon(x - vt), \varepsilon^2 t)e^{ik_c x + i\omega_c t}\varphi_{k_c} + \mathrm{c.c.} \tag{27}$$

with $v = \left(\frac{d(\mathrm{Im}\,\lambda_1)}{dk}\right)\Big|_{k=k_c}$, $\omega_c = \mathrm{Im}\,\lambda_1|_{k=k_c}$, and $\varphi_{k_c} = \varphi_1(k_c) \in \mathbb{C}^3$ leads to the complex Ginzburg–Landau equation

$$\partial_T A = c_1 A + c_2 \partial_X^2 A - c_3 A|A|^2 \tag{28}$$

with coefficients $c_j \in \mathbb{C}$ and complex-valued amplitude $A = A(X,T)$. It has been shown rigorously [Schn99] that certain aspects of the Taylor–Couette problem can be approximated by the complex Ginzburg–Landau equation.

Notations: We denote the space of n-times weakly differentiable local uniformly Sobolev functions with $H^n_{l,u}$. This Banach space is equipped with the norm

$$||u||_{H^n_{l,u}} = \sup_{x\in\mathbb{R}} ||u(\cdot)||_{H^n(x,x+1)}.$$

For details we refer to [Schn99]. This is the space in which the initial reduction to the complex Ginzburg-Landau equation is carried out.

Theorem 5.1. *For all* C_1, $T_0>0$ *there exist* C_2, $\varepsilon_0>0$ *such that for all* $\varepsilon\in(0,\varepsilon_0)$ *the following is true. Let* $A\in C([0,T_0],H^3_{l,u})$ *with*

$$\sup_{T\in[0,T_0]} ||A(T)||_{H^3_{l,u}} < C_1$$

be a solution of the complex Ginzburg–Landau equation (28). *Then there exist solutions* U *of the Taylor–Couette problem* (25) *with*

$$\sup_{t\in[0,\frac{T_0}{\varepsilon^2}]} ||U(t) - \Upsilon(t)||_{H^2_{l,u}} \leqslant C_2\varepsilon^2,$$

where $\Upsilon(t)$ *is defined by the right-hand side of* (27).

Proof. See [Schn99]. □

Combining Theorem 4.1 with Theorem 5.1 gives.

Theorem 5.2. *For all* $m\geqslant 1$, $p\in\mathbb{N}$, $\tau_0>0$, *and* $\rho>0$ *there exist* $C_1>0$, $C_2>0$, $\tau_1>0$, *and* $\delta_0>0$ *such that for all* $\delta\in(0,\delta_0)$ *the following holds. Let* $\psi\in C([0,\tau_0],\mathscr{X}^0_{2\rho})$ *be a solution of the conservation law* (26) *associated to the complex Ginzburg–Landau equation* (28) *with*

$$\sup_{\tau\in[0,\tau_0]} ||\psi(\tau)||_{\mathscr{X}^0_{2\rho}} \leqslant C_1$$

and let $(\mathbf{s}^*,\Psi^*)$ *be the improved approximation constructed in the proof of Theorem* 3.1 *with approximation rate* $\mathcal{O}(\delta^p)$. *Then there exist* $\varepsilon_0>0$ *and* $C_3>0$ *such that for all* $\varepsilon\in(0,\varepsilon_0)$ *we have solutions* $U=U(\tau)$ *of the Taylor–Couette problem in PRII such that for all* $r\in(0,p)$

$$\sup_{t\in[0,\frac{\tau_1}{\varepsilon^2\delta}]} \; \sup_{x\in[-(\varepsilon\delta)^{-r},(\varepsilon\delta)^{-r}]} \Big| U(x,t) - \varepsilon(1+\mathbf{s}^*(\psi)(\varepsilon\delta(x-vt),\varepsilon^2\delta t))$$

$$\times \exp\left(i \int_0^{\varepsilon\delta(x-vt)} \Psi^*(\delta X', \varepsilon^2 \delta t)\, \mathrm{d}X' + i\varepsilon^2\omega_0 t + i\omega_c t + \phi(0,T) + ik_c x \right) - \text{c.c.} \Bigg|$$
$$\leqslant C_2 \varepsilon \delta^{p-r} + C_3 \varepsilon^2.$$

It is the purpose of further research to prove such an approximation result also for $\varepsilon > 0$ not small.

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